PATH-BY-PATH LQG CONTROL OF DISCRETE-TIME SWITCHED AND MARKOVIAN JUMP LINEAR SYSTEMS

JI-WOONG LEE†, GEIR E. DULLERUD‡, AND PRAMOD P. KHARGONEKAR§

Abstract. LQG control problems for guaranteeing finite-horizon performance subject to stability are proposed for switched linear systems and Markovian jump linear systems in the discrete-time domain. Exact, convex synthesis conditions for dynamic output feedback controllers are expressed in terms of nested unions of linear matrix inequalities. Thus feedback solutions are obtained by solving semidefinite programs offline. The resulting controllers recall a finite number of past modes of the plant’s operation. Moreover, closed-loop stability does not require long control horizon.

Key words. discrete linear inclusion, linear matrix inequality (LMI), Pareto optimality, receding horizon control

AMS subject classifications. 49N10, 93B12, 90C22, 93C55

1. Introduction. Switched systems and Markovian jump systems model multi-modal systems under nondeterministic switching between different modes of operation. A switched system consists of a finite number of state-space models, each of which corresponds to a mode of operation, and a set of admissible switching sequences among these models [30, 29, 38, 19, 31]. A Markovian jump system, on the other hand, is in a sense a refinement of a switched system, where admissible switching sequences are the realizations of a Markov chain [32, 9]. These systems appear in many contexts such as networked control systems [21, 6, 40, 15], macroeconomic models [12, 39], distributed networks of autonomous vehicles [17, 33], and biological and chemical processes [2, 35]. Following the approach taken in [24, 23, 27, 28], we propose and solve novel LQG control problems for these systems with linear state-space models.

In this paper, we focus on a certain Pareto optimal controller synthesis problem. We consider finite-horizon performance measures over a forward window of length $T$. Our objective is to design a controller that assures internal stability of the closed-loop system. It is allowed to have a finite memory of the most recent length-$M$ switching path of modes (where $M$ is computed as part of the solution) as well as perfect observation of the present mode. Our control objective is to minimize the worst-case $T$-step receding-horizon cost over all start times and over all admissible switching sequences. Instead of optimizing a single uniform performance bound, we seek to achieve Pareto-optimal path-by-path output regulation, which leads to a novel result in control of switched linear systems and Markovian jump linear systems.

As shown in Fig. 1.1, the lengths $T$ and $M$ can be considered the sizes of the forward window and the backward window, respectively. While the forward window $T$ is a given length, the backward window $M$ is to be determined via offline semidefinite programming. Even though finite-horizon optimization for a given horizon length involves looking ahead to future switching paths, the designed controller coefficients depend only on the past switching path of a finite length. This ensures the controller is

---

*Early conference versions of this paper were presented at the IEEE Conference on Decision and Control in 2007 and 2008.
†Department of Electrical Engineering, Pennsylvania State University, University Park, PA 16802 (jiwoong@psu.edu).
‡Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801 (dullerud@illinois.edu).
§Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611 (ppk@ufl.edu).
causal and also guarantees (almost sure) uniform exponential stability of the closed-loop system. We note that the closed-loop system is guaranteed to be stable by design regardless of the length $T$. Also, the number of matrix variables in the offline semidefinite design program is independent of $T$; in particular, it does not grow as $T$ increases.

This problem formulation resembles that of receding-horizon control, where a finite-horizon online optimization is performed at each time step over a moving horizon window. Receding horizon control is useful in approximately solving hard infinite-horizon optimal control problems, for online optimization–based model predictive control under state-control constraints, and when short-term optimization is emphasized over infinite-horizon planning [20, 14, 7, 22]. Previous results on receding horizon–type control of Markovian jump systems and switched systems (see, e.g., [12, 13, 34, 1]) require a sufficiently long control horizon to guarantee closed-loop stability. Moreover, these results are restricted to so-called mode-dependent controllers, which depend only on the (estimated) current mode of operation, even though controllers with memory of past modes are known to outperform these controllers [10, 28].

In summary, the novel features of the proposed problem and corresponding solution are as follows:

- By focusing on path-by-path Pareto-optimality in the root-mean-square value of the error output, one can give different weights to different switching paths (cf. optimization of path-by-path root-mean-square gain in [23]);
- Closed-loop uniform exponential stability is guaranteed and does not require a sufficiently long horizon length $T$ (cf. references cited above);
- The number of matrix variables in the offline semidefinite program does not depend on $T$ (see the comment at the end of Section 3.1);
- Dynamic output feedback controllers are allowed to use a (finite) memory of past modes (cf. mode-dependent and Markov control laws in [5, 11, 16, 10]).

Limiting cases of our results capture known previous results. For example, the case where the horizon length $T$ is zero reduces to the peak output variance minimization problem considered in [27]. On the other hand, while the average output variance minimization problem (i.e., the infinite-horizon LQG problem) in [27] is restricted to the case where admissible switching sequences are defined by a directed graph, our approach with a sufficiently long horizon length can in principle be used to approximately solve the infinite-horizon LQG problem under arbitrary sets of admissible switching sequences.

Sections 2 and 3 present the definitions needed to analyze the performance of switched and Markovian jump systems and the main analysis results, respectively. Likewise, Sections 4 and 5 give additional definitions and the main synthesis results, respectively. Numerical examples are provided in Section 6 to illustrate the presented results. Finally, concluding remarks are given in Section 7. In order to maximize readability of the main results of the paper, all the proofs are collected in the Appendices. Early conference versions of this paper appeared in [25, 26].
Notation. The Euclidean vector norm on $\mathbb{R}^n$ is denoted by $\| \cdot \|$ so that $\| x \| = \sqrt{x^T x}$ for $x \in \mathbb{R}^n$. If $X, Y \in \mathbb{R}^{n \times n}$ are symmetric and if $X - Y$ is negative definite (resp. negative semidefinite) than we write $X < Y$ (resp. $X \leq Y$). The trace of an $X \in \mathbb{R}^{n \times n}$ is denoted by $\text{tr} X$. In partitioned matrix representations, we use * to indicate off-diagonal entries that can be deduced from symmetry and also to indicate some blocks on the diagonal whose entries are irrelevant.

2. Definitions for Analysis.

2.1. Switched Linear Systems. Given positive integers $n, m, l, \text{and } N$, let

$$G = \{(A_1, B_1, C_1, D_1), \ldots, (A_N, B_N, C_N, D_N)\}$$

$$\subset \mathbb{R}^{n \times n} \times \mathbb{R}^{l \times m} \times \mathbb{R}^{l \times n} \times \mathbb{R}^{l \times m}$$

be a set of $N$ matrix quadruples, and let

$$\Theta \subset \{1, \ldots, N\}^\infty$$

be a set of infinite sequences in $\{1, \ldots, N\}$. Then the pair $(G, \Theta)$ defines the discrete-time switched linear system, which is the family of state-space representations

$$x(t + 1) = A_{\theta(t)} x(t) + B_{\theta(t)} w(t),$$
$$z(t) = C_{\theta(t)} x(t) + D_{\theta(t)} w(t)$$

over all $(\theta(0), \theta(1), \ldots) \in \Theta$. Infinite sequences with values in $\{1, \ldots, N\}$ shall be called switching sequences and those in $\Theta$ are called admissible switching sequences.

With an admissible switching sequence $\theta = (\theta(0), \theta(1), \ldots)$ given, the switched linear system is said to be in mode $i$ at time $t$ if $\theta(t) = i$. For nonnegative integers $T$ and for $\Theta \subset \{1, \ldots, N\}^\infty$, define

$$W_T(\Theta) = \{(\theta(t), \ldots, \theta(t + T)) : \theta \in \Theta, t = 0, 1, \ldots\}.$$

Our stability notion requires that the state sequence $x = (x(0), x(1), \ldots)$ decreases exponentially and uniformly in time and over all $\theta \in \Theta$ in the absence of disturbance inputs.

**Definition 2.1.** The switched linear system $(G, \Theta)$ is said to be uniformly exponentially stable if there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that, whenever $w = 0$, the state sequence $x$ satisfies

$$\|x(t)\| \leq c \lambda^{t - t_0} \|x(t_0)\|$$

for all $t_0, t \in \{0, 1, \ldots\}$ with $t_0 \leq t$, for all $x(t_0) \in \mathbb{R}^n$, and for all $\theta \in \Theta$.

On the other hand, we assess the performance of a switched linear system by measuring the worst-case average error output variance per unit time over all $\theta \in \Theta$ under white Gaussian disturbance inputs and under the zero initial state. Let $w$ be the i.i.d. sequence of $\mathbb{R}^m$-valued Gaussian random variables satisfying

$$E[\|w(t)\|^2] = 0 \text{ for all } t; \quad E[w(t)w(s)^T] = \begin{cases} I & \text{for } t = s; \\ 0 & \text{for } t \neq s; \end{cases}$$

where $E[\cdot]$ denotes the expectation. If the error output sequence $z$ is generated under a switching sequence $\theta \in \Theta$, then we denote $z = (z_\theta(0), z_\theta(1), \ldots)$. 

DEFINITION 2.2. Let $T$ be a given nonnegative integer, and let

\begin{equation}
\Gamma = \{ \gamma(i_0,\ldots,i_T) : (i_0,\ldots,i_T) \in \mathcal{W}_T(\Theta) \} \subset (0,\infty)
\end{equation}

be an indexed family of positive numbers $\gamma(i_0,\ldots,i_T)$. The switched linear system $(\mathcal{G}, \Theta)$ is said to satisfy $T$-step path-by-path performance levels $\Gamma$ if, whenever $w$ satisfies (2.5) and $x(0) = 0$, the error output sequence $z = (z_0(0), z_0(1), \ldots)$ satisfies

\begin{equation}
\frac{1}{T+1} \sum_{t=t_0}^{t_0+T} \mathbb{E} \| z_\theta(t) \|^2 < \gamma(\theta(t_0),\ldots,\theta(t_0+T))
\end{equation}

for all $t_0 \in \{0,1,\ldots\}$ and for all $\theta \in \Theta$. Moreover, if $\gamma > 0$ is such that $\gamma \geq \gamma(i_0,\ldots,i_T)$ for all $(i_0,\ldots,i_T) \in \mathcal{W}_T(\Theta)$, then $(\mathcal{G}, \Theta)$ is said to satisfy $T$-step uniform performance level $\gamma$.

As $T \to \infty$, a $T$-step uniform performance level approaches the square root of an infinite-horizon LQG performance level [27, Definition 3.4]. On the other hand, if $T = 0$, then it reverts to the square root of a peak output variance level [27, Definition 3.3]. In general, Definition 2.2 gives us a receding-horizon-type performance measure that enables offline synthesis of feedback controllers. The path-by-path performance levels defined here are similar to those in [23, Definition 5.1] in the sense that they are refinements of uniform performance levels. However, while [23, Definition 5.1] is concerned with the root-mean-square “gain” of the system from $w$ to $z$, Definition 2.2 above is about the root-mean-square “value” of $z$ under a white Gaussian noise $w$.

2.2. Markovian Jump Linear Systems. Given a positive integer $N$, let $P = (p_{ij}) \in [0,1]^{N \times N}$ be a row-stochastic matrix such that $\sum_{j=1}^{N} p_{ij} = 1$ for all $i \in \{1,\ldots,N\}$; let $p = (p_i) \in [0,1]^{1 \times N}$ be a row-vector such that $\sum_{i=1}^{N} p_i = 1$. The pair $(P,p)$ then defines a discrete-time homogeneous Markov chain with transition probability matrix $P$ and initial distribution $p$. If we denote the state of Markov chain $(P,p)$ at time $t$ by $\theta(t)$, then each realization $(\theta(0), \theta(1), \ldots)$ of $(P,p)$ is a switching sequence. In particular, define

\begin{equation}
\Theta(P,p) = \{ (\theta(0), \theta(1), \ldots) : p_{\theta(0)}>0, p_{\theta(t)|\theta(t+1)}>0, t = 0, 1, \ldots \},
\end{equation}

and call the switching sequences belonging to $\Theta(P,p)$ admissible with respect to $(P,p)$. Let $P$ be the unique consistent probability measure [37] on $\{1,\ldots,N\}^\infty$ such that

\begin{equation}
P\{\theta(t+1) = j \mid \theta(t) = i\} = p_{ij}; \quad P\{\theta(0) = i\} = p_i
\end{equation}

for all $i,j \in \{1,\ldots,N\}$, and for all $t = 0, 1, \ldots$.

If $(P,p)$ is a Markov chain and if $\mathcal{G}$ is an indexed family of matrix quadruples as in (2.1), then the triple $(\mathcal{G}, P, p)$ defines the discrete-time Markovian jump linear system, whose state-space representation is given by (2.3) for each realization $\theta = (\theta(0), \theta(1), \ldots)$ of $(P,p)$. In this case, the state $\theta(t)$ of the chain $(P,p)$ at time $t$ defines the mode of the system $(\mathcal{G}, P, p)$ at time $t$. As is done in the series of papers [24, 23, 27], our notion of stability for Markovian jump linear systems requires almost sure uniformity in the exponential decay rate of the state.

DEFINITION 2.3. The Markovian jump linear system $(\mathcal{G}, \Theta)$ is said to be almost surely uniformly exponentially stable if there exists a set $\Theta \subset \{1,\ldots,N\}^\infty$ with $P(\Theta) = 1$ such that the switched linear system $(\mathcal{G}, \Theta)$ is uniformly exponentially stable.
Almost surely uniformly exponentially stable Markovian jump linear systems with irreducible $P$ are mean square stable [27] and hence almost surely pointwise exponentially stable [18, 8]. In fact, almost sure uniform exponential stability is a deterministic property that is robust against uncertainties that preserve the sparsity patterns of $P$ and $p$ [24], and so it can be exceedingly conservative for some applications. On the other hand, it is a natural stability notion when, while the sparsity patterns of $P$ and $p$ are known, the individual values of transition probabilities and initial probabilities are uncertain. Moreover, together with a suitable LQG-type performance criterion, it gives us a control problem that complies with the classical stochastic control framework where a stochastic performance measure is combined with the deterministic uniform exponential stability notion. Our performance measure for Markovian jump linear systems is analogous to that for switched linear systems.

**Definition 2.4.** Given a nonnegative integer $T$ and a positive number $\gamma$, the Markovian jump linear system $(\mathcal{G}, P, p)$ is said to satisfy $T$-step average performance level $\gamma$ if, whenever $w$ satisfies (2.5) and $x(0) = 0$, the error output sequence $z$ satisfies

$$\frac{1}{T+1} \sum_{t=t_0}^{t_0+T} \mathbf{E} \|z(t)\|^2 < \gamma^2$$

for all $t_0 \in \{0, 1, \ldots\}$, where $\mathbf{E}$ denotes the expectation with respect to both $w$ and $\theta$.

Throughout the paper, we assume for simplicity that the initial distribution $p$ is invariant under the transition probability matrix $P$ (i.e. $pP = p$) and say $p$ is $P$-invariant; however, we do not assume irreducibility of $P$. If $p$ is $P$-invariant, the $T$-step average performance level is equivalent to the limit of the square root of the average of the error output variance within a finite horizon window $(t_0, \ldots, t_0 + T)$ of length $T$ as the start time $t_0$ approaches infinity, and such a limit exists if $(\mathcal{G}, P, p)$ is almost surely uniformly exponentially stable. This is in contrast to the infinite-horizon LQG performance measure in [27, Definition 5.4], where the average of the error output variance over the entire time line $(0, 1, \ldots)$ is taken. Thus, Definition 2.4 provides a receding-horizon-type LQG performance criterion that, as will be seen below, enables offline performance optimization subject to almost sure uniform exponential stability.

3. Analysis Results.

3.1. Analysis of Switched Linear Systems. Let $\mathcal{G}$ and $\Theta$ be as in (2.1) and (2.2), respectively. For any nonnegative integer $T$ and indexed set $\Gamma$ of positive numbers as in (2.6), we give an exact convex condition for switched linear systems to satisfy $T$-step path-by-path performance levels $\Gamma$.

To simplify notation, write $\theta(t) = 0$ for all $t < 0$ whenever $\theta \in \Theta$. For any given nonnegative integer $L$, define

$$\mathcal{L}_L(\Theta) = \{ (\theta(t-L), \ldots, \theta(t)) : \theta \in \Theta, t = 0, 1, \ldots \}.$$ 

While $W_T(\Theta)$ defined in (2.4) is the set of switching paths of length $T$ that can occur in $\Theta$ starting from time $t$ over all $t \in \{0, 1, \ldots\}$, the set $\mathcal{L}_L(\Theta)$ collects switching paths of length $L$ that may have occurred in $\Theta$ up to time $t$ over all $t \in \{0, 1, \ldots\}$. Removing from $\mathcal{L}_L(\Theta)$ all the switching paths that contain a 0 would result in $W_L(\Theta)$. We shall use the convention that $(j_0, \ldots, j_l)$ is a switching path of length $l - k$ if $l \geq k$ and that $(j_0, \ldots, j_l) = 0$ otherwise.
Theorem 3.1. Given a nonnegative integer $T$ and an indexed family $\Gamma$ of positive numbers $\gamma(i_0, \ldots, i_T)$ as in (2.6), the switched linear system $(G, \Theta)$ is uniformly exponentially stable and satisfies $T$-step path-by-path performance levels $\Gamma$, if and only if there exist a nonnegative integer $M$ and matrices $Y(j_0, \ldots, j_{M-1}) > 0$, $(j_0, \ldots, j_M) \in L_M(\Theta)$, such that

\[(3.2a) \quad A_{i_M} Y(j_0, \ldots, j_{M-1}) A_{i_M}^T - Y(i_1, \ldots, i_M) < -B_{i_M} B_{i_M}^T \]

for all $(i_0, \ldots, i_M) \in W_M(\Theta)$, and such that

\[(3.2b) \quad \frac{1}{T+1} \sum_{t=M}^{M+T} \text{tr} \left( C_n Y((i_{t-M}, \ldots, i_t) - C_n^T + D_n D_n^T) \right) < \gamma^2((i_M, \ldots, i_{M+T}) \in \mathcal{W}_T(\Theta). \]

\[ \text{Proof.} \text{ See Appendix A.} \]

In the case of a single mode where $N = 1$, inequalities (3.2) collapse to the standard Lyapunov-type linear matrix inequalities. Setting $\gamma(i_0, \ldots, i_T) = \gamma$ for some $\gamma > 0$ and for all $(i_0, \ldots, i_T) \in W_T(\Theta)$ in Theorem 3.1 leads to an exact analysis of $T$-step uniform performance as well. While the exact infinite-horizon LQG performance analysis given by [27, Theorem 3.7] is restricted to the cases where $\Theta$ admits a directed-graph representation, this $T$-step uniform performance analysis with a sufficiently long horizon length $T$ provides an approximate infinite-horizon LQG performance analysis for switched linear systems without such a restriction on $\Theta$. On the other hand, Theorem 3.1 leads to an exact analysis of the peak output variance level [27, Theorem 3.6] when $T = 0$. In general, Theorem 3.1 with a relatively short horizon length $T$ gives us an exact short-term performance analysis of uniformly exponentially stable switched linear systems. This theorem can also be used to obtain the least upper bound on a convex combination of $T$-step path-by-path performance levels by running a semidefinite program that minimizes such a convex combination subject to (3.2).

The path lengths $M$ and $T$ play distinct roles in Theorem 3.1. Due to the undecidable problem nature [3, 4], the backward length $M$ cannot be determined at the outset but should be allowed to be free for nonconservative analysis of stability. Fixing $M = 0$ or $M = 1$ corresponds to the conservative assumption that a common or mode-dependent quadratic Lyapunov function exists [5]; if $M = 0$ or $M = 1$ does not work, one has the option to pay additional computational cost and try $M \geq 2$ in return for less conservative performance analysis. On the other hand, the forward length $T$ is fixed and reflects one’s emphasis on short-term performance. While the number of matrix variables in (3.2) grows exponentially in $M$ in the worst case, it does not depend on $T$. Moreover, if one is interested in a single uniform performance bound, then the number of inequalities (3.2) does not depend on $T$ as well.

3.2. Analysis of Markovian Jump Linear Systems. Let $(P, p)$ be a Markov chain with $P = (p_{ij})$ and $p = (p_i)$. Let $G$ and $\Theta(P, p)$ be as in (2.1) and (2.8), respectively. We will identify the $T$-step average performance level of a Markovian jump linear system $(G, P, p)$ as a convex combination of the $T$-step path-by-path performance levels of the corresponding switched linear system $(G, \Theta(P, p))$, where the coefficients for the convex combination are the $T$-step probabilities of the Markov chain $(P, p)$.

If $p$ is $P$-invariant and if switching sequences $\theta$ are realizations of $(P, p)$, then, for each switching path $(i_0, \ldots, i_T)$ of length $T$, the probability that $(\theta(t), \ldots, \theta(t+T)) =
representations controlled switched linear system the discrete-time be a set of
\[(4.2)\]
\[u \text{ } K \text{ } n\] a finite memory of past modes. For nonnegative integers
\[t\] observed by the controller at each time \[T\]
\[\text{short-term performance optimization, subject to stability, over long-term planning.}\]
\[\text{Theorem 5.8} \] can be approximated by choosing a sufficiently long horizon length and with (3.4) added to (3.2). The infinite-horizon LQG performance analysis in [27, (3.3)]
\[\pi \text{ } (i_0, \ldots, i_T) \in W_T(\Theta(P, p))\]
\[\text{Theorem 3.2. Given a nonnegative integer } T, \text{ the Markovian jump linear system (G, P, p), where p is P-invariant, is almost surely uniformly exponentially stable and satisfies T-step average performance level } \gamma > 0 \text{ if and only if the switched linear system (G, } \Theta(P, p)) \text{ is uniformly exponentially stable and satisfies T-step path-by-path performance levels } \gamma_{(i_0, \ldots, i_T)} > 0, (i_0, \ldots, i_T) \in W_T(\Theta), \text{ such that}\]
\[\sum_{(i_0, \ldots, i_T) \in W_T(\Theta(P, p))} \pi_{(i_0, \ldots, i_T)} \gamma_{(i_0, \ldots, i_T)}^2 \leq \gamma^2,\]
where \[\pi_{(i_0, \ldots, i_T)}\] are the T-step probabilities given by (3.3).
\[\text{Proof. See Appendix B.} \]
Due to Theorem 3.2, an exact analysis of the T-step average performance of Markovian jump linear systems is given by Theorem 3.1 with \(\Theta\) replaced by \(\Theta(P, p)\)
and with (3.4) added to (3.2). The infinite-horizon LQG performance analysis in [27, Theorem 5.8] can be approximated by choosing a sufficiently long horizon length \(T\) and minimizing \(\gamma\). On the other hand, using a short horizon length \(T\) will emphasize short-term performance optimization, subject to stability, over long-term planning. As in the case of switched systems, the number of matrix variables required to compute the T-step average performance does not depend on \(T\).

4.1. Closed-Loop Switched Linear Systems. Given positive integers \(n, m_1, m_2, l_1, l_2, \text{ and } N,\) let
\[(4.1)\]
\[T = \{(A_i, B_{1,i}, B_{2,i}, C_{1,i}, D_{11,i}, D_{12,i}, C_{2,i}, D_{21,i}): \text{ } i = 1, \ldots, N\} \subset \mathbb{R}^{n \times m_1} \times \mathbb{R}^{n \times m_2} \times \mathbb{R}^{l_1 \times n} \times \mathbb{R}^{l_1 \times m_1} \times \mathbb{R}^{l_2 \times m_2} \times \mathbb{R}^{l_2 \times m_1}\]
be a set of \(N\) matrix tuples, and let \(\Theta\) be as in (2.2). Then the pair \((T, \Theta)\) defines the discrete-time controlled switched linear system, which is the family of state-space representations
\[\begin{align*}
x(t + 1) &= A_{\theta(t)} x(t) + B_{1,\theta(t)} w(t) + B_{2,\theta(t)} u(t), \\
z(t) &= C_{1,\theta(t)} x(t) + D_{11,\theta(t)} w(t) + D_{12,\theta(t)} u(t), \\
y(t) &= C_{2,\theta(t)} x(t) + D_{21,\theta(t)} w(t)
\end{align*}\]
over all admissible switching sequences \(\theta = (\theta(0), \theta(1), \ldots) \in \Theta,\) where \(u = (u(0), u(1), \ldots)\) is the control input sequence and \(y = (y(0), y(1), \ldots)\) the measured output sequence.

We assume that, in addition to the measured output \(y(t),\) the mode \(\theta(t)\) is perfectly observed by the controller at each time \(t.\) Moreover, the controller is allowed to have a finite memory of past modes. For nonnegative integers \(n_K\) and \(L,\) let
\[(4.3)\]
\[K = \{(A_{K,(i_0, \ldots, i_L)}, B_{K,(i_0, \ldots, i_L)}, C_{K,(i_0, \ldots, i_L)}, D_{K,(i_0, \ldots, i_L)}): (i_0, \ldots, i_L) \in L(\Theta)\} \subset \mathbb{R}^{n_K \times n_K} \times \mathbb{R}^{n_K \times l_2} \times \mathbb{R}^{m_2 \times n_K} \times \mathbb{R}^{m_2 \times l_2}\]
be a set of matrix quadruples whose cardinality is equal to that of $\mathcal{L}_L(\Theta)$, which is defined in (3.1). Set $\theta(t) = 0$ for $t < 0$ and for $\theta \in \Theta$. Define

$$\theta_L(t) = (\theta(t - L), \ldots, \theta(t))$$

for $\theta \in \Theta$, and

$$\Theta_L = \{ (\theta_L(0), \theta_L(1), \ldots) : \theta \in \Theta \}.$$ 

Then the pair $(K, \Theta_L)$ is identified with the L-path-dependent, or finite-path-dependent, (linear dynamic output feedback) controller of order $n_K$ represented by

$$x_K(t + 1) = A_{K, \theta_L}(t)x_K(t) + B_{K, \theta_L}(t)y(t),$$

$$u(t) = C_{K, \theta_L}(t)x_K(t) + D_{K, \theta_L}(t)y(t)$$

for all $t \in \{0, 1, \ldots \}$ and $\theta \in \Theta$. Note that the controller coefficients are constrained to depend solely on the current mode and the most recent $L$ past modes; this ensures that the controller can be used under nondeterministic autonomous switching sequences.

If we write $\hat{x}(t) = [x(t)^T x_K(t)^T]^T$ for all $t$, the feedback interconnection of the controlled plant (4.2) and the finite-path-dependent controller (4.5) is described by

$$\hat{x}(t + 1) = \hat{A}_{\theta_L}(t)\hat{x}(t) + \hat{B}_{\theta_L}(t)w(t),$$

$$\dot{z}(t) = \hat{C}_{\theta_L}(t)\hat{x}(t) + \hat{D}_{\theta_L}(t)w(t)$$

for each $\theta_L = (\theta_L(0), \theta_L(1), \ldots) \in \Theta_L$, where

$$\hat{A}_{(i_0, \ldots, i_L)} = \hat{A}_{i_L} + \hat{B}_{2,i_L}K_{(i_0, \ldots, i_L)}\hat{C}_{2,i_L},$$

$$\hat{B}_{(i_0, \ldots, i_L)} = \hat{B}_{1,i_L} + \hat{B}_{2,i_L}K_{(i_0, \ldots, i_L)}\hat{D}_{21,i_L},$$

$$\hat{C}_{(i_0, \ldots, i_L)} = \hat{C}_{1,i_L} + \hat{D}_{12,i_L}K_{(i_0, \ldots, i_L)}\hat{C}_{2,i_L},$$

$$\hat{D}_{(i_0, \ldots, i_L)} = \hat{D}_{11,i_L} + \hat{D}_{12,i_L}K_{(i_0, \ldots, i_L)}\hat{D}_{21,i_L},$$

with

$$K_{(i_0, \ldots, i_L)} = \begin{bmatrix} A_{K,(i_0, \ldots, i_L)} & B_{K,(i_0, \ldots, i_L)} \\ C_{K,(i_0, \ldots, i_L)} & D_{K,(i_0, \ldots, i_L)} \end{bmatrix}$$

for $(i_0, \ldots, i_L) \in \mathcal{L}_L(\Theta)$, and with

$$\hat{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B}_{1,i} = \begin{bmatrix} B_{1,i} \\ 0 \end{bmatrix}, \quad \hat{B}_{2,i} = \begin{bmatrix} 0 & B_{2,i} \end{bmatrix},$$

$$\hat{C}_{1,i} = \begin{bmatrix} C_{1,i} \\ 0 \end{bmatrix}, \quad \hat{D}_{12,i} = \begin{bmatrix} 0 & D_{12,i} \end{bmatrix},$$

$$\hat{C}_{2,i} = \begin{bmatrix} 0 & I \\ C_{2,i} & 0 \end{bmatrix}, \quad \hat{D}_{21,i} = \begin{bmatrix} 0 \\ D_{21,i} \end{bmatrix}$$

for $i \in \{1, \ldots, N\}$. Here, $\hat{x} = (\hat{x}(0), \hat{x}(1), \ldots)$ is the closed-loop state sequence, and each $\theta_L \in \Theta_L$ is an admissible closed-loop switching sequence. If $\theta_L(t) = (i_0, \ldots, i_L)$ for some $t \in \{0, 1, \ldots \}$ and some $\theta_L \in \Theta$, then the closed-loop system (4.6) is said to be in closed-loop mode $(i_0, \ldots, i_L)$. That is, the set of all closed-loop modes is equal to the set $\mathcal{L}_L(\Theta)$ whenever the controller is L-path-dependent. If we put

$$T_K = \{ (\hat{A}_{(i_0, \ldots, i_L)}, \hat{B}_{(i_0, \ldots, i_L)}, \hat{C}_{(i_0, \ldots, i_L)}, \hat{D}_{(i_0, \ldots, i_L)}) : (i_0, \ldots, i_L) \in \mathcal{L}_L(\Theta) \},$$
then the pair \((T_k, \Theta_L)\) defines the closed-loop switched linear system, whose state-space representation is (4.6) for each \(\theta_L \in \Theta_L\).

**Definition 4.1.** Given a nonnegative integer \(T\) and an indexed family \(\Gamma\) of positive numbers \(\gamma_{(t_0,\ldots,t_T)}\) as in (2.6), the \(L\)-path-dependent linear dynamic output feedback controller \((K, \Theta_L)\) is said to be \(\Gamma\)-admissible for \((T, \Theta)\) if the closed-loop switched linear system \((T_k, \Theta_L)\) is uniformly exponentially stable and satisfies \(T\)-step path-by-path performance levels \(\gamma\).

### 4.2. Closed-Loop Markovian Jump Linear Systems

If \(T\) is as in (4.1) and if \((P, p)\) is a Markov chain with transition probability matrix \(P\) and initial distribution \(p\), then the triple \((T, P, p)\) defines the discrete-time controlled Markovian jump linear system, whose state-space representation is the same as (4.2) over all realizations \(\theta = (\theta(0), \theta(1), \ldots)\) of \((P, p)\).

Let \(\Theta(P, p)\) be as in (2.8). For nonnegative integers \(L\), let \(K\) and \(T_k\) be as in (4.3) and (4.7), respectively, with \(\Theta\) replaced by \(\Theta(P, p)\). Then, as in [27], we can define a state transition matrix \(Q_L(P, p) = (q_{(i_0,\ldots,i_L)})\) for the switching paths in \(L(L(\Theta(P, p))\) as follows: \(q_{(i_0,\ldots,i_L)}(j_0,\ldots,j_L) = p_{i_0,j_0}\) if \((i_1,\ldots,i_L) = (j_0,\ldots,j_{L-1})\), and \(q_{(i_0,\ldots,i_L)}(j_0,\ldots,j_L) = 0\) otherwise; also, \(q_{(i_0,\ldots,i_L)} = p_{i_L}\) if \((i_0,\ldots,i_{L-1}) = (0,\ldots,0)\), and \(q_{(i_0,\ldots,i_L)} = 0\) otherwise. Then the feedback interconnection of the controlled Markovian jump linear system \((T, P, p)\) and an \(L\)-path-dependent controller \((K, \Theta(P, p))\) yields a closed-loop Markovian jump linear system \((T_k, Q_L(P, p), q_L(P, p))\), where the pair \((Q_L(P, p), q_L(P, p))\) defines the closed-loop Markov chain. The state-space representation of \((T_k, Q_L(P, p), q_L(P, p))\) is given by (4.6) for all realizations \(\theta_L\) of \((Q_L(P, p), q_L(P, p))\).

**Definition 4.2.** Given a nonnegative integer \(T\) and a positive number \(\gamma\), the \(L\)-path-dependent linear dynamic output feedback controller \((K, \Theta_L(P, p))\) is said to be \((\gamma, T)\)-admissible for \((T, P, p)\) if the closed-loop Markovian jump linear system \((T_k, Q_L(P, p), q_L(P, p))\) is almost surely uniformly exponentially stable and satisfies \(T\)-step average performance level \(\gamma\).

### 5. Synthesis Results

#### 5.1. Synthesis of Switched Linear Systems

Let \(T\) and \(\Theta_L\) be as in (4.1) and (4.4). In this section, we present an exact, convex synthesis condition for finite-path-dependent controllers that deliver \(T\)-step path-by-path performance levels for a switched linear system. The result is based on the exact analysis given by Theorem 3.1.

**Theorem 5.1.** Given nonnegative integers \(n_K \geq n\) and \(T\), and an indexed family \(\Gamma\) of positive numbers \(\gamma_{(i_0,\ldots,i_T)}\) as in (2.6), there exists a \(\Gamma\)-admissible finite-path-dependent linear dynamic output feedback controller of order \(n_K\) for \((T, \Theta)\) if and only if there exist a nonnegative integer \(M\), symmetric matrices \(R_{(i_0,\ldots,i_{M-1})} \in \mathbb{R}^{n \times n}\), \(S_{(j_0,\ldots,j_{M-1})} \in \mathbb{R}^{n \times n}\), \(Z_{(j_0,\ldots,j_M)} \in \mathbb{R}^{l_1 \times l_1}\), and rectangular matrices \(W_{(j_0,\ldots,j_M)} \in \mathbb{R}^{(n+m_2)(n+l_2)}\) such that

\[
\begin{align*}
(H_{(i_0,\ldots,i_M)} + P^T F_{(i_0,\ldots,i_M)} G_{i_M} + G_{i_M}^T W_{(i_0,\ldots,i_M)}^T F_{i_M} & < 0, \\
(\hat{H}_{(i_0,\ldots,i_M)} + \hat{F}_{(i_0,\ldots,i_M)} G_{i_M} + G_{i_M}^T W_{(i_0,\ldots,i_M)}^T \hat{F}_{i_M} & < 0
\end{align*}
\]

for all \((i_0,\ldots,i_M) \in L_M(\Theta),\) and such that

\[
\frac{1}{T+1} \sum_{t=M}^{M+T} \text{tr} Z_{(i_{t-M},\ldots,i_t)} < \gamma_{(i_M,\ldots,i_{M+T})}^2
\]
for all \((i_0, \ldots, i_{M+T}) \in \mathcal{L}_{M+T}(\Theta)\) with \((i_M, \ldots, i_{M+T}) \in \mathcal{W}_T(\Theta)\), where

\[
(5.2a) \quad H_{(i_0, \ldots, i_M)} = \begin{bmatrix}
-S_{(i_0, \ldots, i_{M-1})} & -I & A_T^{i_M} & R_{(i_0, \ldots, i_{M-1})}A_T^{i_M} & -R_{(i_1, \ldots, i_M)}A_T^{i_M} \\
* & * & * & * & * \end{bmatrix}
\]

\[
(5.2b) \quad G_{i_M} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\
C_{2,i_M} & 0 & 0 & 0 & D_{21,i_M} \end{bmatrix}
\]

\[
(5.2c) \quad F_{i_M} = \begin{bmatrix} 0 & 0 & 0 & I & 0 \\
0 & 0 & B_{2,i_M}^T & 0 & 0 \end{bmatrix}
\]

\[
(5.2d) \quad \tilde{H}_{(i_0, \ldots, i_M)} = \begin{bmatrix}
-S_{(i_0, \ldots, i_{M-1})} & -I & C_{1,i_M}^T & R_{(i_0, \ldots, i_{M-1})}C_{1,i_M}^T & -Z_{(i_0, \ldots, i_M)} & D_{11,i_M} & -I \\
* & * & * & * & * & * & * \\
\end{bmatrix}
\]

\[
(5.2e) \quad \tilde{G}_{i_M} = \begin{bmatrix} 0 & I & 0 & 0 \\
C_{2,i_M} & 0 & 0 & D_{21,i_M} \end{bmatrix}
\]

\[
(5.2f) \quad \tilde{F}_{i_M} = \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & D_{12,i_M} & 0 \end{bmatrix}
\]

for all \((i_0, \ldots, i_M) \in \mathcal{L}_M(\Theta)\). Moreover, such a controller can be taken to be \(M\)-path-dependent and of order \(n\).

**Proof.** See Appendix C. \(\square\)

One way to jointly optimize the path-by-path performance levels \(\gamma_{(i_0, \ldots, i_T)}\) over all \(T\)-paths \((i_0, \ldots, i_T) \in \mathcal{W}_T(\Theta)\) is to minimize a convex combination of these levels. The path-by-path performance levels resulting from this optimization will be Pareto optimal; that is, no path-by-path performance levels \(\tilde{\gamma}_{(i_0, \ldots, i_T)}\) satisfying \(\tilde{\gamma}_{(i_0, \ldots, i_T)} \leq \gamma_{(i_0, \ldots, i_T)}\) for all \((i_0, \ldots, i_T) \in \mathcal{W}_T(\Theta)\) and \(\tilde{\gamma}_{(j_0, \ldots, j_T)} < \gamma_{(j_0, \ldots, j_T)}\) for some \((j_0, \ldots, j_T) \in \mathcal{W}_T(\Theta)\) are achievable subject to closed-loop stability.

Given a (forward) horizon length \(T\), a (backward) path length \(M\), and an indexed family \(\Gamma\) of \(T\)-step path-by-path performance levels \(\gamma_{(i_0, \ldots, i_T)}\), \((i_0, \ldots, i_T) \in \mathcal{W}_T(\Theta)\), under which the synthesis condition in Theorem 5.1 is satisfied, the coefficients of a \(\Gamma\)-admissible \(M\)-path-dependent controller \((\mathcal{K}, \Theta_L)\) can be obtained as in \([36, 27]\): If \(R_{(j_0, \ldots, j_{M-1})}, S_{(j_0, \ldots, j_{M-1})}, Z_{(j_0, \ldots, j_{M-1})}\), and \(W_{(j_0, \ldots, j_M)}\) solve linear matrix inequalities (5.1), then obtain \(A_K(i_0, \ldots, i_M), B_K(i_0, \ldots, i_M), C_K(i_0, \ldots, i_M),\) and \(D_K(i_0, \ldots, i_M)\) by solving the linear algebraic equation

\[
W_{(i_0, \ldots, i_M)} = \begin{bmatrix} S_{(i_0, \ldots, i_M)}A_{i_M} & R_{(i_0, \ldots, i_{M-1})} & 0 \\
0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} U_{(i_1, \ldots, i_M)} & S_{(i_0, \ldots, i_{M-1})}B_{2,i_M} \\
0 & 0 & 0 \end{bmatrix}
\]

\[
\times \begin{bmatrix} A_{K,(i_0, \ldots, i_M)} & B_{K,(i_0, \ldots, i_M)} & C_{K,(i_0, \ldots, i_M)} & D_{K,(i_0, \ldots, i_M)} \\
T_{(i_0, \ldots, i_{M-1})} & 0 \end{bmatrix}
\]

for all \((i_0, \ldots, i_M) \in \mathcal{L}_M(\Theta)\), where \(T_{(j_0, \ldots, j_{M-1})}\) and \(U_{(j_0, \ldots, j_{M-1})}\) are any nonsingular
5.2. Synthesis of Markovian Jump Linear Systems. Let \( T \) be as in (4.1), and let \((P, p)\) be a Markov chain. An exact, convex synthesis condition for finite-path-dependent controllers that deliver a \( T \)-step average performance level for a Markovian jump linear system is provided in this section. The result is based on the switched output variance per unit time for each \( \text{switched linear system} \). Let \( \Theta \) be \( P \)-invariant. Given nonnegative integers \( n_K \geq n \) and \( T \) and given a positive number \( \gamma \), there exists a \((\gamma, T)\)-admissible finite-path-dependent linear dynamic output feedback controller of order \( n_K \) for \((T, P, p)\) if and only if there exist a nonnegative integer \( M \), symmetric matrices \( R_{(j_0, \ldots, j_{M-1})} \in \mathbb{R}^{n \times n} \), \( S_{(j_0, \ldots, j_{M-1})} \in \mathbb{R}^{n \times n} \), \( Z_{(j_0, \ldots, j_{M})} \in \mathbb{R}^{n \times n} \), and rectangular matrices \( W_{(j_0, \ldots, j_{M})} \in \mathbb{R}^{n \times (n+1)} \) such that (5.1a) and (5.1b) hold for all \((i_0, \ldots, i_M)\) \( \in L_M(\Theta(P, p)) \), and such that (5.1c) holds for all \((i_0, \ldots, i_{M+T})\) \( \in L_{M+T}(\Theta(P, p)) \) with \((i_M, \ldots, i_{M+T})\) \( \in W_T(\Theta(P, p)) \), so that

\[
\sum_{(i_M, \ldots, i_{M+T}) \in W_T(\Theta(P, p))} \pi_{(i_M, \ldots, i_{M+T})} \gamma^2_{(i_M, \ldots, i_{M+T})} \leq \gamma^2,
\]

where \( \pi_{(i_M, \ldots, i_{M+T})} \) are as in (3.3). Moreover, such a controller can be taken to be \( M \)-path-dependent and of order \( n \).

Proof. See Appendix D. \( \square \)

In view of Theorem 5.2, running the semidefinite program that minimizes \( \gamma^2 \) subject to linear matrix inequalities (5.1) and (5.4) yields a Pareto optimum at which a convex combination of path-by-path performance levels for the associated switched linear system is at minimum. Conversely, such Pareto optimization leads to closed-loop stability and the optimal average performance level for a Markovian jump linear system if the coefficients for the convex combination are the \( T \)-step probabilities of the underlying Markov chain.

6. Illustrative Examples. This section presents three numerical examples. Examples 1 and 2 give analysis and synthesis of switched linear systems, and Example 3 gives synthesis of a Markovian jump linear system.

Example 1. Let \( n = m = l = 1 \) and \( N = 2 \). Let \( G \) in (2.1) have

\[
A_1 = 1/\sqrt{2}, \quad B_1 = 1, \quad C_1 = 1, \quad D_1 = 0;
A_2 = 1/\sqrt{2}, \quad B_2 = 1, \quad C_2 = 0, \quad D_2 = 0,
\]

and let \( \Theta \) in (2.2) consist of two periodic switching sequences \((1, 2, 1, 2, \ldots)\) and \((2, 1, 2, 1, \ldots)\) where modes 1 and 2 alternate. Then, as shown in [27, Example 1], the switched linear system \((G, \Theta)\) is uniformly exponentially stable, and has a unit average output variance per unit time for each \( \theta \in \Theta \). On the other hand, the infimum \( \gamma^* \) of the system’s \( T \)-step uniform performance levels for each horizon length \( T \) is given by

\[
\gamma^* = \begin{cases} 
\sqrt{\frac{T+2}{T+1}}, & \text{if } T \text{ is even}; \\
1, & \text{if } T \text{ is odd}.
\end{cases}
\]
An alternative approach would be to first minimize the sum of the performance levels. However, performance optimization over normal operating conditions is desired; that is, over two switching paths (1, 1, . . . , 1) and (3, 3, . . . , 3). One way to address this problem is to run a semidefinite program to minimize a weighted sum of 14 performance levels \( \gamma_{(1,1,1,1,1)}, \gamma_{(1,1,1,1,2)}, \ldots, \gamma_{(2,1,2,1,2),} \gamma_{(3,3,3,3,3)}, \) with large weights for the first and last, subject to (5.1) for \( M = 1. \) (Again, we observe that \( M = 1 \) suffices for optimality.) The resulting performance levels will be Pareto-optimal. An alternative approach would be to first minimize the sum of \( \gamma_{(1,1,1,1,1)} \)

<table>
<thead>
<tr>
<th>( T )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>20</th>
<th>500</th>
<th>\ldots</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma^* )</td>
<td>1.4143</td>
<td>1.1548</td>
<td>1.0955</td>
<td>1.0236</td>
<td>1.0050</td>
<td>1.0010</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

In this particular example, applying Theorem 3.1 with \( M = 0 \) suffices to obtain the numerical results shown in Table 6.1, which coincide with the analytical values.

**Example 2.** Let \( T \) in (4.1) have \( N = 3 \) and

\[
\begin{align*}
A_1 &= 0.5, \quad B_{1,1} = [1 \ 0], \quad B_{2,1} = 0, \\
C_{1,1} &= [1 \ 0], \quad D_{11,1} = [0 \ 0], \quad D_{12,1} = [0 \ 1], \quad C_{2,1} = 1, \quad D_{21,1} = [1 \ 0]; \\
A_2 &= 1, \quad B_{1,2} = [1 \ 0], \quad B_{2,2} = 1, \\
C_{1,2} &= [1 \ 0], \quad D_{11,2} = [0 \ 0], \quad D_{12,2} = [0 \ 1], \quad C_{2,2} = 1, \quad D_{21,2} = [0 \ 1]; \\
A_3 &= 0.5, \quad B_{1,3} = [1 \ 0], \quad B_{2,3} = 0, \\
C_{1,3} &= [1 \ 0], \quad D_{11,3} = [0 \ 0], \quad D_{12,3} = [0 \ 1], \quad C_{2,3} = 1, \quad D_{21,3} = [0 \ 1].
\end{align*}
\]

Writing \( ij \) for \( (i, j), i, j \in \{1, 2, 3\} \), let

\[
\Theta = \{((\theta(0), \theta(1), \ldots) \in \{1, 2, 3\}^\infty; \theta(0) \in \{1, 3\}, \theta(t)\theta(t + 1) \in \{11, 12, 21, 33\} \text{ for all } t \geq 0\}.
\]

Then, according to [27, Example 5], the optimal achievable uniform infinite-horizon output regulation level for \( (T, \Theta) \) is approximately 1.240, and there exists a one-path-dependent controller that achieves this performance level. For each horizon length \( T \), let \( \gamma_L \) denote the infimum of the finite-step uniform performance levels obtained from Theorem 5.1 with \( M = L \). Table 6.2 shows that, for all odd horizon lengths \( T \), a one-path-dependent controller delivers a \( T \)-step uniform performance level of 1.240, which is the same as the infinite-horizon performance level. On the other hand, for even horizon lengths \( T \), we have \( \gamma_L = \gamma_1 \) for all \( L \geq 1 \) and \( T \geq 0 \), and the optimal finite-step uniform performance level \( \gamma^* \) approaches its infinite-horizon counterpart. Applying the result of [27] yields that the one-path dependent controller that delivers a ten-step uniform performance level of 1.247 (as shown in Table 6.2) achieves an infinite-horizon performance level of 1.240, which is essentially optimal.

Now, let us fix \( T = 4 \) and suppose that mode 2 corresponds to a rare event associated with a fault in the system. Closed-loop stability must be guaranteed under occasional faults. However, performance optimization over normal operating conditions is desired; that is, over two switching paths (1, 1, . . . , 1) and (3, 3, . . . , 3). One way to address this problem is to run a semidefinite program to minimize a weighted sum of 14 performance levels \( \gamma_{(1,1,1,1,1)}, \gamma_{(1,1,1,1,2)}, \ldots, \gamma_{(2,1,2,1,2),} \gamma_{(3,3,3,3,3)}, \) with large weights for the first and last, subject to (5.1) for \( M = 1. \) (Again, we observe that \( M = 1 \) suffices for optimality.) The resulting performance levels will be Pareto-optimal. An alternative approach would be to first minimize the sum of \( \gamma_{(1,1,1,1,1)} \)
Numerical results of Example 2. The optimal $T$-step uniform performance level $\gamma^*$ is achieved with $M = 1$ for all $T$, and converges to the infinite-horizon performance level as $T$ increases.

<table>
<thead>
<tr>
<th>$T$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>$\ldots$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>1.495</td>
<td>1.409</td>
<td>1.441</td>
<td>1.409</td>
<td>1.428</td>
<td>1.409</td>
<td>1.423</td>
<td>1.409</td>
<td>1.420</td>
<td>$\ldots$</td>
<td>N/A</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>1.262</td>
<td>1.240</td>
<td>1.259</td>
<td>1.240</td>
<td>1.253</td>
<td>1.240</td>
<td>1.250</td>
<td>1.240</td>
<td>1.248</td>
<td>$\ldots$</td>
<td>N/A</td>
</tr>
<tr>
<td>$\gamma^*$</td>
<td>1.262</td>
<td>1.240</td>
<td>1.259</td>
<td>1.240</td>
<td>1.253</td>
<td>1.240</td>
<td>1.250</td>
<td>1.240</td>
<td>1.248</td>
<td>$\ldots$</td>
<td>1.240</td>
</tr>
</tbody>
</table>

**Fig. 6.1.** The inverted pendulum in Example 3 with three modes of operation. The center of gravity of the pendulum jumps among three locations at each nonnegative-integer time instants according to a Markov chain.

and $\gamma_{(3,3,3,3,3)}$ while fixing the remaining 12 performance levels at a large number and then minimize a uniform bound on the remaining performance levels while fixing $\gamma_{(1,1,1,1,1)}$ and $\gamma_{(3,3,3,3,3)}$ at optimum. Using this alternative approach, we obtain that the optimal values for $\gamma_{(1,1,1,1,1)}$ and $\gamma_{(3,3,3,3,3)}$ are both equal to 1.155 and that the remaining performance levels, all associated with mode 2, are no larger than 1.256. The optimal one-path-dependent controller uses the previous mode, as well as the current mode, to determine its coefficients. If the previous and current modes are $\theta(t-1)$ and $\theta(t)$, then the coefficients are

$$
K_{(\theta(t-1),\theta(t))} = 
\begin{bmatrix}
A_{K_{(\theta(t-1),\theta(t))}} & B_{K_{(\theta(t-1),\theta(t))}} \\
C_{K_{(\theta(t-1),\theta(t))}} & D_{K_{(\theta(t-1),\theta(t))}}
\end{bmatrix},
$$

where

$$
K_{(1,1)} = 
\begin{bmatrix}
-0.4069 & -2.516 \\
0 & 0
\end{bmatrix},
K_{(1,2)} = 
\begin{bmatrix}
0.3011 & -0.1152 \\
0.1760 & -0.06736
\end{bmatrix},
K_{(2,1)} = 
\begin{bmatrix}
-0.2914 & -2.135 \\
0 & 0
\end{bmatrix},
\text{and } K_{(3,3)} = 
\begin{bmatrix}
0.1805 & -0.5731 \\
0 & 0
\end{bmatrix}.
$$

**Example 3.** An inverted pendulum with three modes of operation is shown in Fig. 6.1. A pendulum of mass $M_p$ is mounted upside down on a moving cart of mass $M_c$, which is free to move on a straight frictionless track. The moment of inertia of the pendulum is $J_p$, and the gravitational acceleration $g$. Let $\phi$ be the angular displacement of the pendulum rod from the vertical, and let $\rho$ be the horizontal displacement of the cart from the origin. In mode 1, the angle $\phi$ between the vertical and the center of gravity of the pendulum equals $\varphi$. However, in mode 2 (resp.
mode 3) we have \( \phi = \varphi - 45^\circ \) (resp. \( \phi = \varphi + 45^\circ \)). In all modes, the distance between the pivot and the center of gravity of the pendulum remains the same and is equal to \( L_p \). Assuming that the mode switching sequence \( \theta \) is a realization of a Markov chain \( (\mathbf{P}, \mathbf{p}) \), which jumps at time instants \( t = 0, 1, \ldots \) with \( \mathbf{p} \) being \( \mathbf{P} \)-invariant, and that \( \varphi, \rho, \) and \( \theta \) are perfectly observed, our objective is to apply force \( \mu \) to the cart to maintain the angular displacement \( \varphi \) near \( 0^\circ \) under random disturbances and noisy measurements.

The equations of motion for the inverted pendulum is given by

\[
(J_p + M_p L_p^2)\ddot{\varphi} = M_p g L_p \sin \phi - M_p L_p \dot{\rho} \cos \phi,
\]

\[
(M_p + M_c)\ddot{\rho} = \mu + M_p L_p \rho^2 \sin \phi - M_p L_p \dot{\rho} \cos \phi.
\]

After linearizing these equations about the solution

\[
(\varphi(t), \dot{\varphi}(t), \rho(t)) = (\phi_{\theta(t)}, 0, (M_p + M_c) g \tan \phi_{\theta(t)}),
\]

where \( \phi_1 = 0, \phi_2 = -45^\circ, \) and \( \phi_3 = 45^\circ, \) and then zero-order-hold sampling the result with unity sampling period, we obtain a controlled Markovian jump linear system with \( N = 3 \) and

\[
A_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -M_p \rho & 0 & 0 & -M_p L_p \cos \phi_i \\ 0 & -M^2 g L^2 & 0 & 0 \\ -M^2 g L^2 \Xi_i & 0 & -M_p L_p \rho \cos \phi_i & 0 \end{bmatrix}, \quad B_{2,i} = \begin{bmatrix} 0 \\ -M_p L_p \cos \phi_i \\ 0 \\ J_p + M_p L_p^2 \Xi_i \end{bmatrix}, \quad C_{2,i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]

where \( \Xi_i = (M_p + M_c)(J_p + M_p L_p^2) - M^2 g L^2 \cos^2 \phi_i \) for \( i \in \{1, 2, 3\} \). (We are not concerned with the error of approximation with respect to the linearized models.) Letting \( M_p = 0.5, M_c = 1, J_p = 0.01, L_p = 1, g = 9.8, \) and

\[
\mathbf{P} = \begin{bmatrix} 1/8 & 3/4 & 1/8 \\ 1/4 & 3/4 & 0 \\ 0 & 3/4 & 1/4 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 3/4 & 1/8 & 1/8 \end{bmatrix},
\]

and assuming

\[
B_{1,i} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_{1,i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
D_{11,i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_{12,i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_{21,i} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix},
\]

for all \( i \in \{1, 2, 3\} \), we obtain the results shown in Table 6.3, where \( \gamma_L \) denotes the infimum of the finite-step average performance levels obtained from Theorem 5.2 with \( M = L \); again, for this particular example, we have \( \gamma_L = \gamma_1 \) for all \( L \geq 1 \) and \( T \geq 0 \). In summary, for each horizon length \( T \), minimizing the right-hand side of (5.4) subject to (5.1) for path length \( M = 1 \) yields an optimal controller for this example.

7. Concluding Remarks. A novel approach to path-by-path Pareto-optimal control of switched linear systems and Markovian jump linear systems was proposed. One merit of our approach is that the control horizon does not need to be long enough.
Table 6.3

Numerical results of Example 3. The optimal $T$-step average performance level $\gamma^*$ is achieved with $M = 1$ for all $T$, and converges to the infinite-horizon performance level as $T$ increases.

<table>
<thead>
<tr>
<th>$T$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>410</td>
<td>410</td>
<td>410</td>
<td>410</td>
<td>...</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>116</td>
<td>107</td>
<td>104</td>
<td>102</td>
<td>101</td>
<td>...</td>
<td>N/A</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$\gamma^*$</td>
<td>116</td>
<td>107</td>
<td>104</td>
<td>102</td>
<td>101</td>
<td>...</td>
<td>95.8</td>
</tr>
</tbody>
</table>

... to assure stability. In addition, the synthesis condition for dynamic output feedback controllers is necessary and sufficient. For performance optimization, this approach involves semidefinite programming over future switching paths of a given length $T$; on the other hand, to guarantee stability, the semidefinite program is subject to a set of Lyapunov inequalities over past switching paths of length $M$. While the number of matrix variables in the semidefinite program depends directly on $M$, it does not depend on $T$. As the backward length $M$ approaches infinity, the achievable performance bound approaches its optimal value for a given forward length $T$. However, saturation in performance often occurs at a small $M$, and hence, in practice, the path length $M$ can be taken to remain constant over all $T$.

**Appendix A. Proof of Theorem 3.1.**

The proof requires a linear matrix inequality–based characterization of uniformly exponentially stable switched linear systems.

**Lemma A.1.** The switched linear system $(G, \Theta)$ is uniformly exponentially stable if and only if there exist a nonnegative integer $M$ and symmetric positive definite matrices $Y_{(i_0, \ldots, i_M-1)} \in \mathbb{R}^{n \times n}$ such that

$$A_{i_M} Y_{(i_0, \ldots, i_M-1)} A_{i_M}^T - Y_{(i_1, \ldots, i_M)} < 0$$

for all $(i_0, \ldots, i_M) \in \mathcal{W}_M(\Theta)$.

**Proof.** Combining [23, Theorem 3.3] and [23, Corollary 3.4] yields that the system $(G, \Theta)$ is uniformly exponentially stable if and only if $A_{i_M}^T X_{(i_1, \ldots, i_M)} A_{i_M} - X_{(i_0, \ldots, i_M-1)} < 0$ for some $M$ and $X_{(j_0, \ldots, j_{M-1})} > 0$, and for all $(i_0, \ldots, i_M) \in \mathcal{W}_M(\Theta)$. Then putting $Y_{(j_0, \ldots, j_{M-1})} = X_{(j_0, \ldots, j_{M-1})}^{-1}$ for all $(j_0, \ldots, j_M) \in \mathcal{W}_M(\Theta)$ gives the desired result. $\Box$

Let $Y_{\theta,t_0}^{(t_0)} \geq 0$ be the unique solution to the Lyapunov equation

$$A_{\theta(t)} Y_{\theta,t}^{(t_0)} A_{\theta(t)}^T - Y_{\theta,t+1}^{(t_0)} = -B_{\theta(t)} B_{\theta(t)}^T$$

for $\theta \in \Theta$ and for $t \geq t_0$, with $Y_{\theta,t_0} = 0$. If $\Phi_{\theta}(t, t_0)$ is the state transition matrix defined by

$$\Phi_{\theta}(t, t_0) = \begin{cases} I, & t = t_0; \\ A_{\theta(t-1)} \cdots A_{\theta(t_0)}, & t > t_0, \end{cases}$$

then we have

$$Y_{\theta,t+1}^{(t_0)} = \sum_{s=t_0}^{t} \Phi_{\theta}(t+1, s+1) B_{\theta(s)} B_{\theta(s)}^T \Phi_{\theta}(t+1, s+1)^T \quad (A.1)$$
and
\[
(A.2) \quad E \left\| z_\theta(t) \right\|^2 = \text{tr} \left( C_{\theta(t)} Y_{\theta,t}^{(0)} C_{\theta(t)}^T + D_{\theta(t)} D_{\theta(t)}^T \right)
\]
for \( t \geq t_0 \geq 0 \) and \( \theta \in \Theta \). The following lemma is useful in proving the theorem.

**Lemma A.2.** Let \( Y_{t+1} \) be as in (A.1). Then
(a) \( \forall Y_{\theta,t+1} \leq Y_{\theta,t+1}^{(0)} \) for \( t \leq t_0 \).
(b) \( \forall Y_{\theta,t+1} \geq Y_{\theta,t+1}^{(0)} \) for \( t \leq t_0 \), whenever \( Y_{\theta,t_0} \geq 0 \) and
\[
A_{\theta(t)} Y_{\theta,t} A_{\theta(t)}^T - Y_{\theta,t+1} \leq -B_{\theta(t)} B_{\theta(t)}^T
\]
for \( t \leq t_0 \).

**Proof.** The result is immediate from definitions. \( \square \)

Now we are ready to prove the theorem. To show the necessity part, suppose that \( (G, \Theta) \) is uniformly exponentially stable and that \( \gamma_{(i_0, \ldots, i_T)} > 0 \), \( (i_0, \ldots, i_T) \in W_T(\Theta) \), are such that (2.7) holds for all \( \theta \in \Theta \) and \( t \geq 0 \). For \( \varepsilon > 0 \), consider the perturbed system \( (S^\varepsilon, \Theta) \), where
\[
S^\varepsilon = \{(A_i, B_{i}^{(\varepsilon)}, C_i, D_{i}^{(\varepsilon)}) : i = 1, \ldots, N\},
\]
\[
B_{i}^{(\varepsilon)} = [B_i \sqrt{\varepsilon} I] \in \mathbb{R}^{n 	imes (m+n)}, \quad D_{i}^{(\varepsilon)} = [D_i, 0] \in \mathbb{R}^{l 	imes (m+n)}.
\]
If \( Y_{\theta,t}^{(\varepsilon,t_0)} \), \( t \geq t_0 \geq 0 \), are such that
\[
A_{\theta(t)} Y_{\theta,t}^{(\varepsilon,t_0)} A_{\theta(t)}^T - Y_{\theta,t+1}^{(\varepsilon,t_0)} = -B_{\theta(t)}^{(\varepsilon)} B_{\theta(t)}^{(\varepsilon) T}
\]
for \( \theta \in \Theta \) and \( t \geq t_0 \), with \( Y_{\theta,t_0}^{(\varepsilon,t_0)} = 0 \), then
\[
Y_{\theta,t+1}^{(\varepsilon,t_0)} = \sum_{s=t_0}^{t} \Phi_{\theta}(s + 1, s + 1)(B_{\theta(s)} B_{\theta(s)}^T + \varepsilon I) \Phi_{\theta}(t + 1, s + 1)^T.
\]

Hence there exists a sufficiently small \( \varepsilon > 0 \) and positive numbers \( v_{(i_0, \ldots, i_T)} \) for \( (i_0, \ldots, i_T) \in W_T(\Theta) \) such that
\[
(A.3a) \quad v_{(\theta(t_0), \ldots, \theta(t_0+T))} = \gamma_{(\theta(t_0), \ldots, \theta(t_0+T))} - \varepsilon > 0,
\]
\[
(A.3b) \quad A_{\theta(t)} Y_{\theta,t}^{(\varepsilon,t_0)} A_{\theta(t)}^T - Y_{\theta,t+1}^{(\varepsilon,t_0)} = -B_{\theta(t)}^{(\varepsilon)} B_{\theta(t)}^{(\varepsilon) T} - \varepsilon I,
\]
\[
(A.3c) \quad \frac{1}{T+1} \sum_{t=t_0}^{t_0+T} \text{tr} \left( C_{\theta(t)} Y_{\theta,t}^{(\varepsilon,0)} C_{\theta(t)}^T + D_{\theta(t)} D_{\theta(t)}^T \right) \leq v_{(\theta(t_0), \ldots, \theta(t_0+T))}^2
\]
whenever \( t \geq t_0 \geq 0 \) and \( \theta \in \Theta \). In particular, it follows from (A.3b) that there is a sufficiently large integer \( M > 0 \) such that
\[
(A.4) \quad A_{\theta(t)} Y_{\theta,t}^{(\varepsilon,t-M)} A_{\theta(t)}^T - Y_{\theta,t+1}^{(\varepsilon,t-M+1)} < -B_{\theta(t)}^{(\varepsilon)} B_{\theta(t)}^{(\varepsilon) T}
\]
for all \( t \geq M \) and all \( \theta \in \Theta \). Put
\[
Y_{(\theta(t-M), \ldots, \theta(t-1))}^{(\varepsilon,t)} = \begin{cases} Y_{\theta,t}^{(\varepsilon,t-M)} &, t \geq M; \\ Y_{\theta,t}^{(\varepsilon,0)} &, t < M. \end{cases}
\]
for all \( t \geq 0 \) and \( \theta \in \Theta \) (where \( \theta(t) = 0 \) for \( t < 0 \)). Then (A.4) leads to (3.2a) for all \((i_0, \ldots, i_M) \in \mathcal{W}_M(\Theta)\). By part (a) of Lemma A.2, we have \( Y_{(\theta(t-M), \ldots, \theta(t-1))} \leq Y_{(\theta,t)}^{(x,0)} \) for all \( t \geq 0 \) and \( \theta \in \Theta \), so (A.3a) and (A.3c) imply that (3.2b) holds for all \((i_0, \ldots, i_{M+T}) \in \mathcal{L}_{M+T}(\Theta)\) such that \((i_M, \ldots, i_{M+T}) \in \mathcal{W}_T(\Theta)\).

To show sufficiency, suppose that (3.2a) holds for all \((i_0, \ldots, i_M) \in \mathcal{W}_M(\Theta)\) and (3.2b) for all \((i_0, \ldots, i_{M+T}) \in \mathcal{L}_{M+T}(\Theta)\) such that \((i_M, \ldots, i_{M+T}) \in \mathcal{W}_T(\Theta)\). Then Lemma A.1 yields that the system \((G, \Theta)\) is uniformly exponentially stable. Moreover, part (b) of Lemma A.2 leads to

\[
Y_{\theta,t}^{(x,0)} \leq \begin{cases} Y_0 & \text{if } M = 0; \\
Y_{(\theta(t-M), \ldots, \theta(t-1))} & \text{if } M > 0 
\end{cases}
\]

whenever \( t \geq 0 \) and \( \theta \in \Theta \), and hence we have that the system \((G, \Theta)\) satisfies \( T \)-step path-by-path performance levels \( \gamma_{(i_0, \ldots, i_T)} \). This completes the proof.

**Appendix B. Proof of Theorem 3.2.**

The proof is similar to that of Theorem 3.1, so we will only sketch it. If \((G, P, p)\) is almost surely uniformly exponentially stable, then \((G, \Theta(P, p))\) is uniformly exponentially stable by the following lemma:

**Lemma B.1.** The Markovian jump linear system \((G, P, p)\) is almost surely uniformly exponentially stable if and only if the switched linear system \((G, \Theta(P, p))\) is uniformly exponentially stable.

**Proof.** The result is a special case of \([23, \text{Theorem } 4.3]\). \(\Box\)

Moreover, if \((G, P, p)\) satisfies \( T \)-step average performance level \( \gamma \), then we have

\[
\frac{1}{T+1} \sum_{t=t_0}^{t_0+T} E\|z(t)\|^2 = \sum_{(\theta(t_0), \ldots, \theta(t_0+T))} \frac{\pi(\theta(t_0), \ldots, \theta(t_0+T))}{T+1} \sum_{t=t_0}^{t_0+T} E\|z_\theta(t)\|^2.
\]

Let \( Y_{\theta,t}^{(x,t_0)} \) satisfy (A.3b) for \( \epsilon > 0 \), \( \theta \in \Theta(P, p) \), and \( 0 \leq t_0 \leq t \), subject to the initial condition \( Y_{\theta,t_0}^{(x,t_0)} = 0 \). It can be shown by using (A.1), (A.2), and Lemma A.2, and by proceeding as in the proof of Theorem 3.1 that, whenever the Markovian jump system \((G, P, p)\) satisfies \( T \)-step average performance level \( \gamma \) and the corresponding switched system \((G, \Theta(P, p))\) is uniformly exponentially stable, there exist a sufficiently small number \( \epsilon > 0 \) and a sufficiently large integer \( M > 0 \) such that (A.4) and

\[
\sum_{(\theta(t_0), \ldots, \theta(t_0+T))} \frac{\pi(\theta(t_0), \ldots, \theta(t_0+T))}{T+1} \sum_{t=t_0}^{t_0+T} \text{tr}\left( C_{\theta(t)} Y_{\theta,t}^{(x,t-M)} C_{\theta(t)}^T + D_{\theta(t)} D_{\theta(t)}^T \right) < \gamma^2
\]

hold whenever \( t \geq M \) and \( \theta \in \Theta \). Define \( Y_{(\theta(t-M), \ldots, \theta(t-1))}^{(x,0)} \) as in (A.5) for all \( t \geq 0 \) and \( \theta \in \Theta \). Then one can show that (3.2a) and (3.2b) hold for some set of \( \gamma_{(i_M, \ldots, i_{M+T})} > 0 \), \((i_M, \ldots, i_{M+T}) \in \mathcal{W}_T(\Theta)\), such that (5.4) is satisfied (or equivalently, some set of \( \gamma_{(i_0, \ldots, i_T)} > 0 \), \((i_0, \ldots, i_T) \in \mathcal{W}_T(\Theta)\), such that (3.4) is satisfied). This proves the necessity. Sufficiency follows easily from Lemmas A.2 and B.1.

**Appendix C. Proof of Theorem 5.1.**

Whenever \( L \) and \( M \) are integers such that \( 0 \leq L \leq M \), we define

\[
\begin{align*}
\mathcal{L}_{M-L}(\Theta_L) &= \mathcal{L}_M(\Theta); \\
\mathcal{W}_{M-L}(\Theta_L) &= \{(i_0, \ldots, i_M) \in \mathcal{L}_M(\Theta) : (i_L, \ldots, i_M) \in \mathcal{W}_{M-L}(\Theta)\}.
\end{align*}
\]
It follows from Theorem 3.1 and identity (C.1) that \((\mathcal{K}, \Theta_L)\) is a \(\Gamma\)-admissible synthesis for \((T, \Theta)\) if and only if there exists a nonnegative integer \(M \geq L\) and matrices \(Y_{(j_0, \ldots, j_{M-1})} > 0\) such that

\[
(C.2a) \quad \tilde{A}_{(i_M, \ldots, i_1)} Y_{(i_0, \ldots, i_{M-1})} \tilde{A}^T_{(i_M, \ldots, i_1)} - Y_{(i_1, \ldots, i_M)} < -B_{(i_M, \ldots, i_1)} B^T_{(i_M, \ldots, i_1)}
\]

for all \((i_0, \ldots, i_M) \in \mathcal{W}_{M-L}(\Theta_L)\) and such that

\[
(C.2b) \quad \frac{1}{T+1} \sum_{t=M}^{M+T} \text{tr} \left( \tilde{C}_{(i_t, \ldots, i_0)} Y_{(i_{t-M}, \ldots, i_{t-1})} \tilde{C}^T_{(i_t, \ldots, i_0)} + \tilde{D}_{(i_t, \ldots, i_0)} \tilde{D}^T_{(i_t, \ldots, i_0)} \right) < \gamma^2_{(i_L, \ldots, i_{L+T})}
\]

for all \((i_0, \ldots, i_{M+T}) \in \mathcal{W}_T(\Theta_M)\). A Schur complement argument gives that this condition is equivalent to the requirement that

\[
(C.3a) \quad \begin{bmatrix}
- Y_{(i_0, \ldots, i_{M-1})} & \tilde{A}^T_{(i_M, \ldots, i_1)} & 0 \\
\tilde{A}^T_{(i_M, \ldots, i_1)} & - Y_{(i_1, \ldots, i_M)} & B_{(i_M, \ldots, i_1)} \\
0 & B^T_{(i_M, \ldots, i_1)} & -I
\end{bmatrix} < 0
\]

for all \((i_0, \ldots, i_M) \in \mathcal{W}_{M-L}(\Theta_L)\), and

\[
(C.3b) \quad \begin{bmatrix}
- Y_{(i_0, \ldots, i_{s-1})} & \tilde{C}^T_{(i_s, \ldots, i_{s-1})} & 0 \\
\tilde{C}^T_{(i_s, \ldots, i_{s-1})} & -Z_{(i_{s-M}, \ldots, i_s)} & \tilde{D}_{(i_s, \ldots, i_{s-1})} \\
0 & \tilde{D}^T_{(i_s, \ldots, i_{s-1})} & -I
\end{bmatrix} < 0,
\]

\[
(C.3c) \quad \frac{1}{T+1} \sum_{t=M}^{M+T} \text{tr} Z_{(i_{t-M}, \ldots, i_t)} < \gamma^2_{(i_L, \ldots, i_{L+T})}
\]

for all \(s \in \{M, \ldots, M+T\}\) and for all \((i_0, \ldots, i_{M+T}) \in \mathcal{W}_T(\Theta_M)\). For \((j_0, \ldots, j_M) \in \mathcal{L}_M(\Theta)\), partition \(Y_{(j_0, \ldots, j_{M-1})}^{-1}\) and \(Y_{(j_0, \ldots, j_{M-1})}\) as

\[
(C.4a) \quad Y_{(j_0, \ldots, j_{M-1})}^{-1} = \begin{bmatrix} S_{(j_0, \ldots, j_{M-1})} & U_{(j_0, \ldots, j_{M-1})} \\ U^T_{(j_0, \ldots, j_{M-1})} & * \end{bmatrix},
\]

\[
(C.4b) \quad Y_{(j_0, \ldots, j_{M-1})} = \begin{bmatrix} R_{(j_0, \ldots, j_{M-1})} & T_{(j_0, \ldots, j_{M-1})} \\ T^T_{(j_0, \ldots, j_{M-1})} & * \end{bmatrix},
\]

with \(S_{(j_0, \ldots, j_{M-1})}, R_{(j_0, \ldots, j_{M-1})} \in \mathbb{R}^{n \times n}\) and \(U_{(j_0, \ldots, j_{M-1})}, T_{(j_0, \ldots, j_{M-1})} \in \mathbb{R}^{n \times \kappa}\).

To prove the sufficiency part of the theorem, suppose that (5.1a) and (5.1b) hold for all \((i_0, \ldots, i_M) \in \mathcal{L}_M(\Theta)\), and that (5.1c) holds for all \((i_0, \ldots, i_{M+T}) \in \mathcal{L}_{M+T}(\Theta)\) such that \((i_M, \ldots, i_{M+T}) \in \mathcal{W}_T(\Theta)\); assume \(n_k = n\) without loss of generality. Choose any nonsingular matrices \(T_{(j_0, \ldots, j_{M-1})}, U_{(j_0, \ldots, j_{M-1})} \in \mathbb{R}^{n \times n}\) such that (5.3) holds for all \((j_0, \ldots, j_M) \in \mathcal{L}_M(\Theta)\). Solve

\[
(C.5) \quad W_{(i_0, \ldots, i_M)} = \begin{bmatrix} S_{(i_1, \ldots, i_M)} A_{i_M} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & S_{(i_1, \ldots, i_M)} B_{2,i_M} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{K,(i_M, \ldots, i_1)} & B_{K,(i_M, \ldots, i_1)} \\ C_{K,(i_M, \ldots, i_1)} & D_{K,(i_M, \ldots, i_1)} \end{bmatrix} \begin{bmatrix} T_{(i_0, \ldots, i_{M-1})} & 0 \\ C_{2,i_M} R_{(i_0, \ldots, i_{M-1})} & I \end{bmatrix},
\]
with \( n_K = n \) and \( L = M \), for matrices \( A_K, B_K, C_K \); \( D_K \left( i_0, \ldots, i_M \right) \in \mathcal{L}_M(\theta) \). Reconstruct \( Y_{(i_0, \ldots, i_{M-1})} > 0 \) via (C.4) for all \((j_0, \ldots, j_M) \in \mathcal{L}_M(\theta) \). Then we recover (C.2a) for \( L = M \) and for all \((i_0, \ldots, i_M) \in \mathcal{L}_M(\theta) \), where \( \mathcal{L}_M(\theta) = \mathcal{W}_0(\Theta_M) \); also, we recover (C.2b) for \( L = M \) and for all \((i_0, \ldots, i_{M+T}) \in \mathcal{W}_T(\Theta_M) \). Thus \((K, \Theta_M)\) is a \( \Gamma \)-admissible synthesis for \((T, \Theta)\).

To prove the necessity part of the theorem, suppose that, for some integers \( n_K \geq n \) and \( L \geq 0 \), there exists a \( \Gamma \)-admissible synthesis \((K, \Theta_L)\) of order \( n_K \) for \((T, \Theta)\). Since \((K, \Theta_L)\) is \( L \)-path-dependent, it is \( M \)-path-dependent for all \( M \geq L \), and hence the inequalities in (C.3) and the change-of-variable formula (C.5) lead to the linear matrix inequalities (5.1) after the congruence transformation on inequalities (C.3a) and (C.3b) by

\[
\begin{bmatrix}
M_{Y,(i_0, \ldots, i_{M-1})} & 0 & 0 \\
0 & M_{X,(i_1, \ldots, i_M)} & 0 \\
0 & 0 & I
\end{bmatrix}, \quad \begin{bmatrix}
M_{Y,(i_{-M}, \ldots, i_{-1})} & 0 & 0 \\
0 & I & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

respectively, where

\[
M_{X,(i_1, \ldots, i_M)} = \begin{bmatrix}
I & S_{(i_1, \ldots, i_M)} \\
0 & U_{(i_1, \ldots, i_M)}^T
\end{bmatrix}, \quad M_{Y,(i_0, \ldots, i_{M-1})} = \begin{bmatrix}
I & R_{(i_0, \ldots, i_{M-1})} \\
0 & T_{(i_0, \ldots, i_{M-1})}^T
\end{bmatrix},
\]

for all \((i_0, \ldots, i_M) \in \mathcal{L}_M(\theta) \). This completes the proof.

**Appendix D. Proof of Theorem 5.2.**

Given integers \( L \geq 0 \) and \( T \geq 0 \), define

\[
\mathcal{N}_{L,T,k}(\Theta(P, p)) = \{(i_0, \ldots, i_{L+T}) \in \mathcal{L}_{L+T}(\Theta(P, p)) : (i_0, \ldots, i_{k-1}) = (0, \ldots, 0), (i_k, \ldots, i_{L+T}) \in \mathcal{W}_{L+T-k}(\Theta(P, p)) \}.
\]

for \( k \in \{0, \ldots, L\} \). Then, for \( 0 < k \leq L \), \( \mathcal{N}_{L,T,k}(\Theta(P, p)) \) is the set of all admissible closed-loop switching paths \((\theta(k-L), \ldots, \theta(k+T))\) that can occur with positive probability at time \( t = L - k \); similarly, for \( k = 0 \), \( \mathcal{N}_{L,T,0}(\Theta(P, p)) \) is the set of all admissible \((\theta(t-L), \ldots, \theta(t+T))\) that can occur with positive probability at any given time \( t \geq L \). For each \( k \in \{0, \ldots, L\} \), the \((L + T)\)-step probabilities \( \pi(i_0, \ldots, i_{L+T}) \) over \((i_0, \ldots, i_{L+T}) \in \mathcal{N}_{L,T,k}(\Theta(P, p)) \) are given by

\[
(D.1a) \quad \pi(i_0, \ldots, i_{L+T}) = p_{i_0} p_{i_k i_{k+1}} \cdots p_{i_{L+T-1} i_{L+T}}.
\]

Clearly, we have

\[
(D.1b) \quad \sum_{(i_0, \ldots, i_{L+T}) \in \mathcal{N}_{L,T,k}(\Theta(P, p))} \pi(i_0, \ldots, i_{L+T}) = 1
\]

for each \( k \in \{0, \ldots, L\} \).

We first show that, for any \( T \), \( L \), and \( K \), the closed-loop \( T \)-step probabilities are given by the \((L + T)\)-step probabilities defined in (D.1); that is, the probability of the closed-loop switching path \((\theta_L(t), \ldots, \theta_L(t + T))\) is equal to \( \pi(\theta(t-L), \ldots, \theta(t+T)) \) for all \( t \in \{0, 1, \ldots\} \) and for all realizations \( \theta \) of \((P, p)\) such that \((\theta(t-L), \ldots, \theta(t+T)) \) has positive probability. If \( t = 0 \), then \( \theta_L(0) = (0, \ldots, 0, \theta(0)) \), so \( q_{\theta_L(0)} = p_{\theta(0)} \) under \((Q_L(P, p), q_L(P, p)) \). If \( t = 1 \), then \( \theta_L(1) = (0, \ldots, 0, \theta(0), \theta(1)) \), so the probability of \( \theta_L(1) \) is equal to \( q_{\theta_L(0) q_L(0) \theta(1)} = p_{\theta(0) \theta(0) \theta(1)} \) for each realization \( \theta \) of \((P, p)\).
By induction, we establish that the probability of \((i_0, \ldots, i_L) \in \mathcal{L}_L(\Theta(P, p))\) is given by
\[
p_i \cdot \prod_{k=1}^{L} p_{i_k \rightarrow i_{k+1}}
\]
for some \(k \in \{0, \ldots, L\}\). Thus, whenever \((i_0, \ldots, i_{L+T}) \in \mathcal{L}_{L+T}(\Theta(P, p))\) is such that \((i_L, \ldots, i_{L+T}) \in W_{L+T}(\Theta(P, p))\) and such that
\[
(i_0, \ldots, i_{L+T}) = (0, \ldots, 0, k \text{ times}, i_k, \ldots, i_{L+T})
\]
for some \(k \in \{0, \ldots, L\}\), the \(P\)-invariance of \(p\) yields that the closed-loop \(T\)-step probabilities \(\pi_{(i_0, \ldots, i_{L+T})}\) satisfy (D.1). Finally, the \(P\)-invariance of \(p\) implies that
\[
\sum_{\{i_k, \ldots, i_{L+T} : (i_0, \ldots, i_{L+T}) \in \mathcal{N}_{L+T}(\Theta(P, p))\}} \pi_{(i_0, \ldots, i_{L+T})} = \pi_{(i_L, \ldots, i_{L+T})}
\]
for each \(k \in \{0, \ldots, L\}\).

Now, in view of Theorem 3.2 and the fact that \(T\)-step probabilities are invariant under \(L\)-path-dependent feedback, if we replace \(\Theta\) in Theorem 5.1 with \(\Theta(P, p)\) and add (5.4) to the conditions in Theorem 5.1, then we obtain the desired synthesis result for the controlled Markovian jump linear system \((T, P, p)\).

REFERENCES


